IB Paper 7: Probability
Notes on Examples Paper 4

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Q4: Poisson's equation

We seek a solution to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\cos\frac{\pi x}{2L} \tag{1}$$

(i)

To verify that some function is a solution to the differential equation, we can substitute the function for ϕ in Eqn. (1) and check the result is consistent:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left[A\cos\frac{\pi x}{2L}\right] + \frac{\partial^2}{\partial y^2}\left[A\cos\frac{\pi x}{2L}\right] &= -\cos\frac{\pi x}{2L} && \text{set } \phi = A\cos\frac{\pi x}{2L} \text{ in Eqn. (1)} \\
-A\left(\frac{\pi}{2L}\right)^2\cos\frac{\pi x}{2L} + 0 &= -\cos\frac{\pi x}{2L} && \text{evaluate } x \text{ derivative} \\
\Rightarrow \quad A &= \left(\frac{2L}{\pi}\right)^2 && \text{consistent for this value of } A
\end{aligned}$$

This shows that the given function satisfies the differential equation if $A = \left(\frac{2L}{\pi}\right)^2$.

(ii)

We need to combine the original governing equation and the newly defined ϕ_0, recognising that our chosen value of A makes all terms not involving ϕ_0 sum to zero:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left[\phi_0 + A\cos\frac{\pi x}{2L}\right] + \frac{\partial^2}{\partial y^2}\left[\phi_0 + A\cos\frac{\pi x}{2L}\right] &= -\cos\frac{\pi x}{2L} && \text{set } \phi = \phi_0 + A\cos\frac{\pi x}{2L} \text{ in Eqn. (1)} \\
\frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial y^2} + \frac{\partial^2}{\partial x^2}\left[A\cos\frac{\pi x}{2L}\right] + \frac{\partial^2}{\partial y^2}\left[A\cos\frac{\pi x}{2L}\right] &= -\cos\frac{\pi x}{2L} && \text{split up the derivatives} \\
\frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial y^2} &= 0 && \text{from part (i)} \qquad (2)
\end{aligned}$$

So we have eliminated the right-hand side from Eqn. (1) by expressing it in terms of ϕ_0, which will make solution by separation of variables possible.

Converting the boundary conditions by substitution of $\phi_0 = \phi - A\cos\frac{\pi x}{2L}$:

$$\phi(x = \pm L, y) = 0 \quad \Rightarrow \quad \phi_0(x = \pm L, y) = 0 \qquad \text{because } \cos\pi/2 = 0 \tag{3}$$

$$\phi(x, y = \pm d) = 0 \quad \Rightarrow \quad \phi_0(x, y = \pm d) = -A\cos\frac{\pi x}{2L} \tag{4}$$

(iii)

To separate the variables, we postulate: $\phi_0(x, y) = X(x)Y(y)$ and then:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}[XY] + \frac{\partial^2}{\partial y^2}[XY] &= 0 && \text{substitute our } \phi_0 \text{ into Eqn. (2)} \\
X''Y + XY'' &= 0 && \text{take derivatives} \\
\frac{X''}{X} = -\frac{Y''}{Y} &= c && \text{rearrange} \qquad (5)
\end{aligned}$$

In Eqn. (5), we have two terms that are functions of x only and y set equal. Because x and y are independent variables, and variation of one does not imply change in another, the only way the equation can be valid for all x and y is if both sides evaluate to the same constant.

The separation constant is denoted c and is either positive, negative or zero; the boundary conditions determine which of these possibilities. The only way to work out the sign of the separation constant is to make a guess and look for a contradiction with the boundary conditions. Splitting Eqn. (5) into two ordinary differential equations:

$$X'' - cX = 0 \tag{6}$$

$$Y'' + cY = 0 \tag{7}$$

where the first-order coefficients have opposite signs.

If $c < 0$, then Eqns. (6) and (7) have the standard general solutions:

$$X = B\cos kx + C\sin kx \tag{8}$$

$$Y = D\cosh ky + E\sinh ky \tag{9}$$

where $c = -k^2$. This is what we want, because the $y = \pm d$ boundary condition in Eqn. (4) has a $\cos\frac{\pi x}{2L}$, matching the functional form of Eqn. (8).

If $c > 0$, then Eqns. (6) and (7) have the general solutions:

$$X = B\cosh kx + C\sinh kx \tag{10}$$

$$Y = D\cos ky + E\sin ky \tag{11}$$

where $c = k^2$. This is not what we want, because we cannot create $\phi_0 \propto \cos\frac{\pi x}{2L}$ on the $y = \pm d$ boundary condition by picking values of B and C in Eqn. (10).

If $c = k = 0$, then integrating twice we have linear solutions for X and Y, which again contradict our $\cos\frac{\pi x}{2L}$ boundary condition.

Proceeding as in Eqns. (8) and (9), with $c < 0$ and $c = -k^2$ the complete general solution is the product of X and Y:

$$\phi_0 = XY = [B\cos kx + C\sin kx]\,[D\cosh ky + E\sinh ky] \tag{12}$$

Now to fix the values of B, C, D, and E by applying the boundary conditions. By equating the terms inside cos functions in Eqns. (12) and (4), we see that

$$kx = \frac{\pi x}{2L} \quad \Rightarrow \quad k = \frac{\pi}{2L} \tag{13}$$

Then on $y = +d$

$$\left[B\cos\frac{\pi x}{2L} + C\sin\frac{\pi x}{2L}\right]\left[D\cosh\frac{\pi d}{2L} + E\sinh\frac{\pi d}{2L}\right] = -A\cos\frac{\pi x}{2L} \tag{14}$$

There is no sin term on the right-hand side, so $C = 0$. Equation (14) then becomes

$$B\left[D\cosh\frac{\pi d}{2L} + E\sinh\frac{\pi d}{2L}\right] = -A \tag{15}$$

Proceeding similarly for $y = -d$

$$B\left[D\cosh\frac{\pi d}{2L} - E\sinh\frac{\pi d}{2L}\right] = -A \tag{16}$$

where we have used the fact that $\cosh(-z) = \cosh(z)$ is an even function and $\sinh(-z) = -\sinh(z)$ is an odd function. This is why we chose hyperbolic functions for the Y solution. We could have used exponentials instead, but they lack this symmetry property.

Adding Eqns. (15) and (16):

$$2BD\cosh\frac{\pi d}{2L} = -2A \quad\Rightarrow\quad BD = \frac{-A}{\cosh\frac{\pi d}{2L}} \tag{17}$$

Subtracting Eqns. (15) and (16):

$$2BE\sinh\frac{\pi d}{2L} = 0 \quad\Rightarrow\quad E = 0 \tag{18}$$

We now have the constants we need and ϕ_0 is:

$$\phi_0 = \frac{-A}{\cosh\frac{\pi d}{2L}}\cos\frac{\pi x}{2L}\cosh\frac{\pi y}{2L} \tag{19}$$

Lastly, we convert back to $\phi = \phi_0 + A\cos\frac{\pi x}{2L}$ and put $A = \left(\frac{2L}{\pi}\right)^2$.

$$\phi = \frac{-A}{\cosh\frac{\pi d}{2L}}\cos\frac{\pi x}{2L}\cosh\frac{\pi y}{2L} + A\cos\frac{\pi x}{2L} \tag{20}$$

$$\phi = \left(\frac{2L}{\pi}\right)^2\cos\frac{\pi x}{2L}\left[1 - \frac{\cosh\frac{\pi y}{2L}}{\cosh\frac{\pi d}{2L}}\right] \tag{21}$$